Impact of Storage on the Efficiency and Prices in Real-Time Electricity Markets

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ABSTRACT
We study the effect of energy-storage systems in dynamic real-time electricity markets. We consider that demand and renewable generation are stochastic, that real-time production is affected by ramping constraints, and that market players seek to selfishly maximize their profit. We distinguish three scenarios, depending on the owner of the storage system: (A) the supplier, (B) the consumer, or (C) a stand-alone player. In all cases, we show the existence of a competitive equilibrium when players are price-takers (they do not affect market prices). We further establish that under the equilibrium price process, players’ selfish responses coincide with the social welfare-maximizing policy computed by a (hypothetical) social planner. We show that with storage the resulting price process is smoother than without.

We determine empirically the storage parameters that maximize the players’ revenue in the market. In the case of consumer-owned storage, or a stand-alone storage operator (scenarios B and C), we find that they do not match socially optimal parameters. We conclude that consumers and the stand-alone storage operator (but not suppliers) have an incentive to under-dimension their storage system. In addition, we determine the scaling laws of optimal storage parameters as a function of the volatility of demand and renewables. We show, in particular, that the optimal storage energy capacity scales as the volatility to the fourth power.

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Keywords
Electricity Pricing; Energy Storage System; Market Efficiency; Energy Economics.

1. INTRODUCTION
The process of liberalizing electricity markets is underway worldwide. It amounts to replacing tightly regulated monopolies with lightly regulated competitive markets [28]. Electricity production is managed through scheduling decisions. In a first stage, producers commit to an energy generation schedule determined through forecasts for the demand and generation of renewable energy for the following day. In a second stage, decisions are made in real-time to compensate for forecast errors in load and production. The price volatility in real-time electricity markets\(^1\) raises the question of the efficiency of these markets: Does the selfish behavior of the various actors lead to a socially acceptable situation?

Electricity markets are highly complex dynamical systems. They incorporate renewable energy sources, such as wind and solar, that are highly volatile; loads that are mostly inelastic; and generation units, that are subject to friction and real-time constraints. To avoid blackouts and due to physical constraints, real-time scheduling of energy is critical.

A model of an electricity market that takes into account these dynamical aspects is proposed in [7, 8]. The authors study the competitive equilibriums in a real-time electricity market where demand is stochastic and energy generation is subject to ramping constraints. They show that if all actors are price-takers (they do not affect market prices), then there exists a competitive equilibrium that is efficient: more specifically, the selfish behavior of actors leads indeed to a socially optimal scheduling of generation. However, they also show that the prices that guarantee such an equilibrium exhibit considerable volatility: they oscillate between 0 and a “choke-up” price, and do not concentrate around the marginal production cost. This model has been extended to incorporate network constraints [25, 27], or the presence of renewables [18]; see also [26] for a survey.

Our motivation in this paper is to understand the role of storage in compensating volatility in real-time electricity markets. This is highly relevant for markets with a high penetration of renewables. Exploitation of renewable energy is encouraged in many countries as a means to reduce CO2 emissions. However, renewable energy sources, such as wind and solar photovoltaic, are not dispatchable. A side effect of their high penetration is the increase in the volatility of electricity generation and thus of prices, according to [18]. Therefore, in order to compensate for their volatility, a high penetration of renewables needs to be supported by mechanisms such as storage systems (batteries or pump-hydro) or fast-ramping generators (essentially gas-fired turbines). Storage can be operated by an energy producer, a consumer, or by a stand-alone storage operator. In the last case, the

\(^1\)the peak to mean ratio can be as high as several thousands, e.g. the prices observed in California in 2000-2001 [16].
storage owner needs to generate revenue, hence energy is stored when market prices are low and is provided to the grid when prices are high. We are interested, in particular, in understanding whether the market efficiency results of [26] continue to hold; e.g., is it socially optimal to have stand-alone storage operators that react to real-time prices?

**Contributions.** We extend the wholesale real-time market model of [26] to incorporate a storage system with losses due to the charge/discharge cycles. We consider three scenarios depending on the owner of the storage: (A) the supplier, (B) the consumer, or (C) a stand-alone real-time storage operator. We show that in all three cases, the market is efficient: there exists a price process such that the selfish behaviors of the players coincide with a socially optimal use of the storage and scheduling of the generation. When the storage energy capacity is large, this price process becomes smooth. Moreover, irrespective of the considered scenario, the same decisions concerning bought/sold energy, real-time production, and storage system operation lead to social optimality. These decisions are enforced via the same incentive (pricing) scheme.

We show numerically that when the storage belongs either to the consumer or to a stand-alone real-time storage operator, the storage energy capacity that maximizes the storage owner welfare is strictly smaller than the socially optimal storage energy capacity. Consequently, even though the market is efficient when the storage parameters are fixed, consumers and stand-alone storage operators still have an incentive to under-dimension their storage systems.

Finally, we study the effect of the volatility and of the ramping capabilities of generators on the optimal storage parameters. We show that when the volatility is $\sigma$, the optimal energy capacity of the storage system scales as $\sigma^2$, and its maximum charging/discharging power scales as $\sigma^4$. When the ramping capability of the generators is $\zeta$, the optimal storage energy capacity scales as $1/\zeta^1$, and its maximum charging/discharging power scales as $1/\zeta$. We conclude that a linear increase in the ramping capability of fast-ramping generators (such as gas turbines) entails a cubic decrease in the required energy capacity. In view of the high cost of storage capacity, a paradoxical situation arises: to accommodate a large deployment of renewables and to compensate for the resulting generation volatility, there is an incentive to deploy conventional fast-ramping generators (that have high carbon dioxide emissions) rather than to invest in storage.

**Road-Map.** The rest of the paper is structured as follows. We first describe the market and storage model in Section 3. We study the control problem from a social point of view in Section 4. Section 5 contains the main theoretical results: we show the existence of competitive equilibria in all three scenarios and prove that they are socially optimal. We study the incentives for actors to install storage devices in Section 6. We investigate the relations between volatility, ramping capabilities and optimal storage energy capacity in Section 7. Finally, we conclude in Section 8.

2. RELATED WORK

A large part of the work related to the economic aspects of storage systems investigates optimal energy storage strategies for profit maximization in the electricity market, assuming that storage owner are price-takers [5, 12, 17]. For a wind-farm owner, the authors of [9] study optimal storage strategies in a day-ahead market. Authors often assume that prices are known. The uncertainties due to the price variabiliy and forecasting are also studied in [1, 4].

The economic viability for storage owners is also an important question. On the one hand, it is shown in [24] that there is a strong economic case for storage installations in the New York City region. A similar analysis is conducted for the PJM interconnection (US east-coast) in [20] for which the authors show a moderate storage capacity is viable. On the other hand, according to [22], “storage is not viable from a system perspective until extremely large levels of wind power are seen on the system” in Ireland – a country that envisioned 80% of the electrical consumption generated by wind power plants. Moreover, it is suggested in [15] that pumped-storage hydro-plant operators need to change their business model (currently electricity price arbitrage) because the potential diminishes with the increased penetration of renewables: Inexpensive energy is generated during the noon consumption peak by PV installations.

From a social planner’s perspective, it is shown [21] that the undesired effects of the volatility of renewables can be mitigated via the use of energy storage, with a manageable increase in energy costs, based on a study in the UK. At a European scale, the authors of [2, 14, 23] show how to use model predictive control to update day-ahead production schedule and mitigate energy curtailments. The use of storage can also compensate for forecast uncertainties. Generation scheduling policies that minimize energy losses and the use of fast-ramping generators are developed in [3, 11].

The question of the efficiency of the control by prices of storage devices is raised in a few papers. Using traces of the real market bids data of the French day-ahead market in 2009, the authors of [13] evaluate the storage operation by a public or a private operator by simulation. They show that the public operation leads to higher social welfare and lower cost of supply than the private operation. Using traces from the Swiss market, authors of [6] also show that the optimal control for a price-taker storage owner are socially optimal. However, these two papers only consider a single-run of simulation and do not provide theoretical guarantee.

The authors of [19] go one step further and obtain theoretical guarantees on a model of a purely static setting: loads are predictable on-peak and off-peak periods, and the price depends linearly on the load. They show that for the same three cases of ownership structure as ours (storage belongs to producers, to consumers or to independent actors), the selfish behavior of price-taker agents lead to a socially optimal use of storage. However, when agents influence prices, storage tends to be underused when owned by producers or independent actors or overused if owned by consumers. The reason is that the higher the use of storage is, the smoother the price is. Our model differs greatly from [19] as it incorporates the dynamical aspects of demand and generation.

3. MARKET AND STORAGE MODEL

In this section we first present a model of a two-stage (day-ahead and real-time) electricity market consisting of two main actors, a consumer and a supplier. Both can have access to an energy storage system in the real-time stage of the market, depending on the scenario. We then describe the storage system, and we analyze three scenarios, depending on the owner of the storage system: (A) the supplier, (B) the consumer, or (C) a stand-alone player.
3.1 Two-stage Electricity Markets

We consider an electricity market with two stages: a day-ahead stage and a real-time stage. The two main actors or players in this market are a supplier who produces electricity, and a consumer who buys electricity and serves a fairly large number of end-users.

In the day-ahead market, each day, players forecast a demand profile $d^a(t)$ and plan generation $g^a(t)$ for the next day. The price of electricity is set by market mechanisms to $p^a(t)$, and the consumer agrees to buy an amount of electricity $g^a(t)$ at this price. As often in markets where demand is inelastic and unpredictable, the consumer purchases an extra amount of goods over the predicted demand as a precautionary measure. Hence we assume that $g^a(t) = d^a(t) + r^a$, where $r^a$ is called the fixed reserve. In addition to the conventional energy sources, the supplier uses renewable energy sources (e.g., wind, solar) that provide a volatile production. The forecast of generation of renewables constitutes a fraction of the planned generation $g^a(t)$, and is complemented with conventional energy.

The real-time market deals with situations where the actual demand cannot be met, due to the volatility of the demand and of the production of renewable energy. The actual demand at time $t$ is $D^t(t) = d^a(t) + D(t)$, where $D(t)$ is referred to as the real-time demand, and can be either positive or negative. The renewable forecast error (the difference between the actual and the forecasted production of renewable energy) is denoted by $\Gamma(t)$. The supplier reacts in real-time to unpredictable changes in demand and renewable generation, and she produces an amount of energy $G(t)$ from additional and usually expensive sources.

The total amount of electricity produced at time $t$ is then $G^t(t) = g^a(t) + \Gamma(t) + G(t)$, and the corresponding real-time reserve is defined as $R(t) = G^t(t) - D^t(t)$. The energy produced in real-time is sold at a price $P(t)$ at time $t$. In [8], the authors characterize the competitive equilibria arising in the real-time market — they analyze the amount of real-time energy bought by the consumer, the amount of real-time energy $G(t)$ produced by the supplier, and the price $P(t)$, resulting from the strategic behavior of the market actors.

In this paper, we are interested in studying the effect of the energy bought by the consumer, the amount of real-time or manipulate these prices. The real-time market model deals with situations where the actual demand cannot be met, due to the volatility of the demand and of the production of renewable energy. The actual demand at time $t$ is $D^t(t) = d^a(t) + D(t)$, where $D(t)$ is referred to as the real-time demand, and can be either positive or negative. The renewable forecast error (the difference between the actual and the forecasted production of renewable energy) is denoted by $\Gamma(t)$. The supplier reacts in real-time to unpredictable changes in demand and renewable generation, and she produces an amount of energy $G(t)$ from additional and usually expensive sources.

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In this paper, we are interested in studying the effect of the existence of storage on the real-time market. We make the same assumptions as in [8], i.e.:

(A1) Volatility. We model the forecast errors made in the demand and in the generation of renewable energy, $Z(t) = \Gamma(t) - D(t) + r^a$, as a driftless Brownian motion with volatility $\sigma$, i.e., the variance of $Z(t)$ is $\sigma^2 t$.

(A2) Prices. We assume that the market actors are price takers: the actors strategically define their actions assuming that the prices are exogenous, and that they cannot influence or manipulate these prices.

(A3) Information. We suppose that the market actors share the same information: Before making a decision at time $t$, they have access to the exogenous data up to time $t$ (i.e., $(D(\tau), \Gamma(\tau), P(\tau), \tau \leq t)$ and to all the past decisions of the other players up to time $t$). Mathematically, this assumption means that there exists a filtration $\mathcal{F}_t, t \geq 0$ such that all processes are $\mathcal{F}_t$-adapted.

3.2 Storage Model

We consider a storage system characterized by (i) the maximum amount of energy $B_{\text{max}}$ it can store, i.e., its energy capacity; (ii) the maximum speed $C_{\text{max}}$ at which it can be charged, referred to as the maximum charging power; (iii) the maximum instantaneous amount of energy $D_{\text{max}}$ we can extract from it, i.e., its the maximum discharging power. We model the efficiency of a charge-discharge cycle as follows: only a fraction $\eta \in (0, 1]$ of the injected energy is stored, and the stored energy can be extracted without losses.

More precisely, let $B(t)$ denote the stored energy at time $t$, and let $u(t)$ be the instantaneous energy extracted from the storage. $u(t) < 0$ means that we are currently storing energy. The storage level evolves as follows:

$$\frac{dB}{dt} = -u(t)(\mathbb{1}_{\{u(t) > 0\}} + \eta \mathbb{1}_{\{u(t) < 0\}}).$$ (1)

The storage control process $u = (u(t), t \geq 0)$ satisfies the following constraints: at any time $t$,

$$-C_{\text{max}} \leq u(t) \leq D_{\text{max}},$$ (2)

$$u(t) \geq 0$$ if $B(t) = B_{\text{max}}$, and $u(t) \leq 0$ if $B(t) = 0.$ (3)

We write $u \in X_B$ if (2)-(3) are satisfied. Note that (3), together with (1), is equivalent to $0 \leq B(t) \leq B_{\text{max}}$ for all $t$.

3.3 Scenario A: Storage at the Supplier

We first focus on the case where the supplier controls the storage system. The storage system is used only in the real-time stage of the market. At this stage, deterministic processes describing day-ahead demand, generation, and prices $(d^a(t), g^a(t), p^a(t), t \geq 0)$ are known from the day-ahead market. The real-time market model for this scenario is represented in Figure 1. Next, we describe the strategic decisions that the actors may take, and introduce the notion of dynamic competitive equilibrium.

![Figure 1: Real-time market model in Scenario A: storage is at the supplier.](image-url)
The term $\bar{U}_D(t)$ only contains quantities that cannot be controlled by the consumer in the real-time market. Thus, the quantity of interest is the second term $W_D(t)$. By abuse of notation, we refer to this second term (rather than $U_D(t)$) as the consumer’s payoff.

The consumer’s objective is to maximize her welfare defined as her long-run discounted expected payoff:

$$\max_{E_D \in X_D} W_D := E \int_0^\infty e^{-\gamma t} W_D(t) \, dt.$$  

The expectation is taken over the random process $Z(t)$.

Supplier. The supplier controls three quantities: the real-time energy generation $G(t)$, the storage system via the storage control process $u(t)$, and the energy sold on the real-time market $E_S(t)$. We say that $(E_S, G, u)$ satisfies the constraints of the supplier and we write $(E_S, G, u) \in X_S$ if

- the following ramping constraints for real-time generation are satisfied: $\forall t' > t, \quad \zeta^- \leq \frac{\dot{G}(t') - \dot{G}(t)}{t' - t} \leq \zeta^+$, where $\zeta^- < 0$ and $\zeta^+ > 0$ are called the ramping capabilities;
- the storage constraints are satisfied: $u \in X_B$;
- for all $t \geq 0$, the supplier sells at most $E_S(t) \leq \Gamma(t) + G(t) + u(t)$ on the real-time market.

We denote by $c$ the marginal cost of real-time energy generation. We assume that the energy generated by renewable sources is free. The total generation cost of the conventional energy sources planned in the day-ahead market is equal to $c^a(t)$. The supplier’s payoff at time $t$ is given by:

$$U_S(t) = P(t)E_S(t) + \int_0^t g^a(t')d\zeta^a(t') - cG(t) - c^a(t).$$

As for the consumer, we write $U_S(t) = \bar{U}_S(t) + W_S(t)$, where

$$\bar{U}_S(t) = \int_0^t g^a(t')d\zeta^a(t') - c^a(t),$$

$$W_S(t) = P(t)E_S(t) - cG(t).$$

The term $\bar{U}_S(t)$ contains only quantities that cannot be controlled by the supplier in real-time. Thus, we focus on the second term, $W_S(t)$. Again, by abuse of notation we refer to $W_S(t)$ (rather than $U_S(t)$) as the supplier’s payoff. Observe that the supplier’s utility is increasing in $E_S$. Hence, setting $E_S(t) = \Gamma(t) + G(t) + u(t)$ maximizes her payoff in $X_S$.

The objective of the supplier is to maximize her welfare:

$$\max_{(E_S, G, u) \in X_S} W_S := E \int_0^\infty e^{-\gamma t} W_S(t) \, dt.$$  

Dynamic Competitive Equilibrium. As in Cho and Meyn [8], we introduce the notion of dynamic competitive equilibrium. When the supplier owns and controls the storage, a dynamic competitive equilibrium is defined as follows:

**Definition 1.** (Dynamic competitive equilibrium, storage at the supplier) A dynamic competitive equilibrium is a set of price and control processes $(P^*, E^*_D, E^*_S, G^*, u^*)$ satisfying:

$$E^*_D \in \operatorname{arg\,max}_{E_D \in X_D} W_D,$$  

$$E^*_S, G^*, u^* \in \operatorname{arg\,max}_{(E_S, G, u) \in X_S} W_S,$$  

$$E^*_D = E^*_S.$$  

In the above definition, (4) means that $E^*_D$ constitutes an optimal control from the consumer’s perspective. Similarly, (5) states that $(E^*_S, G^*, u^*)$ is optimal from the supplier’s perspective. Finally, (6) is the market constraint. Note that in (4), the consumer is not subject to the supplier’s constraints and vice-versa for (5). See [8] for a discussion.

### 3.4 Scenario B: Storage at the Consumer

When the consumer has control of the storage, her strategic decisions are represented by the process pair $(E_D, u)$, where $u(t)$ is the storage control process, i.e., the amount of power discharged at time $t$ from the storage.

![Figure 2](image.png)

**Figure 2:** Market model in scenario B: storage is at the consumer.

By abuse of notation we still write the constraints of the consumer as $(E_D, u) \in X_D$. The energy bought is complemented using the storage system, and thus the consumer’s payoff becomes

$$W_D(t) = -(v + c^a)(E_D(t) + u(t) - D(t) + r^d) - P(t)E_D(t).$$

The supplier controls the energy sold $E_S$ and the real-time generation $G$. We say that they satisfy the supplier constraints and we write $(E_S, G) \in X_S$ if and only if $G$ satisfies the ramping constraints and if for all $t \geq 0, 0 \leq E_S(t) \leq G(t) + \Gamma(t)$. The supplier’s payoff remains the same as in the case she controls the storage: $W_S(t) = P(t)E_S(t) - cG(t)$.

In this scenario, the definition of a dynamic competitive equilibrium is similar to that presented in the case the supplier owns the storage: the only difference is that (4) and (5) are replaced respectively by $(E_D, u^*) \in \operatorname{arg\,max}_{E_D, u \in X_D} W_D$ and $(E_S, G^*) \in \operatorname{arg\,max}_{(E_S, G) \in X_S} W_S$. Note that here the expression of $W_D$ is modified compared to (4).

### 3.5 Scenario C: Stand-Alone Storage Operator

In this scenario, the storage is owned by a third player, the stand-alone storage operator who seeks to maximize her profit via arbitrage on the real-time market: buying energy at low prices and selling at high prices.

![Figure 3](image.png)

**Figure 3:** Market model in Scenario C: stand-alone storage operator.

The only control action of the storage operator is $u(t)$, the power discharged from the storage system at time $t$. Her constraints depend on the storage system parameters and are summarized writing $u \in X_B$. The payoff of the storage operator at time $t$ is $W_B(t) = P(t)u(t)$.

The control, constraints, and payoff of the consumer are the same as in Section 3.3. In particular, her payoff is

$$W_D(t) = -(v + c^a)(E_D(t) - D(t) + r^d) - P(t)E_D(t).$$

The control, constraints, and payoff of the consumer are the same as in Section 3.4. Again, the supplier’s payoff remains equal to $W_S(t) = P(t)E_S(t) - cG(t)$.

The definition of a dynamic competitive equilibrium is again similar to the one presented in the case where the
supplier owns the storage. The difference is that (5) is replaced by \(E_S^* \in \arg \max_{E_S,G} \mathbb{E} W_S\), and (6) by \(E_S(t) + u(t) = E_P(t)\). In addition a competitive equilibrium maximizes the welfare of the storage operator: \(u^* \in \arg \max_{u \in \mathcal{X}_B} \mathbb{W}_P\).

4. THE SOCIAL PLANNER’S PROBLEM

In this section we assume that the system is controlled by a single entity, a hypothetical social planner, who decides what is bought and sold, the generation and storage control processes, and whose objective is to maximize the system welfare over all feasible controls. After defining the social planner’s optimal control problem, we characterize its solution, i.e., the socially optimal controls. This analysis will be used later in the paper to state whether the market is efficient, or the strategic behavior of the various market actors has a negative impact in terms of social efficiency.

4.1 Social Welfare

A time \(t\), the total system payoff \(U(t)\) is either defined as \(U(t) = U_D(t) + U_S(t)\) when the storage is controlled by the supplier (or the consumer) or as \(U(t) = U_D(t) + U_S(t) + U_B(t)\) in case the storage is controlled by a stand-alone market actor. It can be easily checked that the total payoff does not depend on the market actor controlling the storage. Let us, for instance, assume that the storage is controlled by the supplier. As previously, we decompose the total payoff into a controllable part \(\bar{W}(t)\) and an uncontrollable part \(\hat{U}(t)\). We have, for any \(t \geq 0\):

\[
\bar{W}(t) = -(v + c^{bo})(E_D(t) - D(t) + r^{da}) - P(t)(E_S(t) - E_D(t) - cG(t)). \tag{7}
\]

When the market constraint \(E_S = E_D\) is further imposed, the controllable part of the total payoff does not depend on the price \(P(t)\) and is an increasing function of \(E_S(t)\). Hence, the socially optimal controls satisfy: \(E_S(t) = E_D(t) = \Gamma(t) + G(t) + u(t)\). These controls must also satisfy the generation and storage constraints, i.e., \((\Gamma + G + u, G, u) \in \mathcal{X}_S\). Recall that the reserve process \(R\) is defined as the difference of the generated energy and of the demand:

\[
R(t) := \Gamma(t) + G(t) - D(t) + r^{da}.
\]

By abusing the notation, we write \((R, u) \in \mathcal{X}_S\) if and only if \((\Gamma + G + u, G, u) \in \mathcal{X}_S\). The controllable part of the total payoff can be further reduced and written as

\[
W(t) = -(v + c^{bo})(R(t) + u(t)) - cR(t). \tag{8}
\]

Compared with \(\bar{W}(t)\), defined in (7), we substracted the uncontrollable term \(c(\Gamma(t) - D(t) + r^{da})\) to obtain \(W(t)\).

A social planner want to maximize the social welfare:

\[
W := \mathbb{E} \int_0^\infty e^{-\gamma t}[-(v + c^{bo})(R(t) + u(t)) - cR(t)]dt.
\]

over all couple of processes \((R, u) \in \mathcal{X}_S\). The solution of this problem is denoted by \((R^*, u^*)\), and the corresponding optimal controls at the various market actors are \(E_D^* = E_S^* = R^* + u^* + D - r^{da}\), \(G^* = R^* + D - \Gamma - r^{da}\), and \(u^*\).

4.2 Socially Optimal Controls

The following theorem characterizes the socially optimal controls. These controls define, for any given reserve \(r\) and storage state \(b\), the way real-time generation evolves, i.e.,

\[
g = \frac{\partial W}{\partial r}\text{, and the storage control process } u\text{. All proofs are presented in Appendix.}
\]

Theorem 1. The socially optimal controls satisfy:

(Generation) For each \(b\), there exists a threshold \(\phi(b)\) such that the socially optimal generation control \(g^*\) is \(\zeta^+\) if \(r < \phi(b)\) and \(\zeta^-\) if \(r > \phi(b)\). \(\phi\) is a nonincreasing function.

(Storage) The optimal storage control \(u^*\) satisfies:

\[
u^* = \begin{cases}
\max(-r, -C^\text{max}) & \text{if } r < 0 \text{ and } b < B^\text{max}, \\
\min(-r, D^\text{max}) & \text{if } r < 0 \text{ and } b > 0, \\
0 & \text{otherwise.}
\end{cases}
\tag{9}
\]

The value function of the social planner problem is:

\[
V(r, b) := \sup_{(R, u) \in \mathcal{X}_S} \mathbb{E}\left[\int_0^\infty e^{-\gamma t} \left[-(v + c^{bo})(R(t) + u(t)) - cR(t)\right]dt\right],
\tag{10}
\]

where \((r, b)\) is the initial condition: \(R(0) = r\) and \(B(0) = b\), and \(\mathbb{E}[\cdot] := \mathbb{E}[\cdot | R(0) = r, B(0) = b]\).

In the proof of the above theorem, we establish that \(\frac{\partial V}{\partial r}\) is well defined (almost everywhere), and that the threshold function \(\phi\) characterizing the optimal generation control is \(\phi(b) = \sup\{r : \frac{\partial V}{\partial r}(r, b) \geq 0\}\). The optimal storage control \(u^*\) has a simple interpretation: When the reserve is positive, the supplier first serves the demand and puts the remaining energy in the storage while ensuring that storage constraints are satisfied (e.g., the charging power cannot exceed \(C^\text{max}\)). If the reserve is negative, the supplier can server only part of the demand using the generated energy and has to extract energy from the storage to serve the remaining demand (if this is at all possible).

Under the socially optimal controls, we denote by \(B^*\) the storage level process. The system dynamics are characterized by \((R^*, B^*)\) and admit a steady-state whose stationary distribution is denoted by \(\pi\). Under the assumption described in Section 3, \(R^*\) is a Brownian motion with variance \(\sigma^2\) and whose average drift varies over time when the generation control \(g^*\) switches values (e.g. from \(\zeta^+\) to \(\zeta^-\)).

4.3 Numerical Example and Energy Units

To illustrate the socially optimal controls defined in Theorem 1, we plot the threshold function \(b \mapsto \phi(b)\) in Figure 4(a) for different values of the storage energy capacity \(B^\text{max}\). We have the storage level \(b\) on the \(x\)-axis and the reserve \(r\) on the \(y\)-axis. Using the same representation, a sample-path of \((R^*, B^*)\) is plotted in Figure 4(b). The sample path starts at \(B^\text{max}(0) = 2.5\) u.e. and \(R(0) = 1\) u.p. The vector field is its drift. It corresponds to the optimal controls \((u^*, g^*)\). The dashed line is the function \(b \mapsto \phi(b)\).

In all figures, the storage energy capacity \(B^\text{max}\) is expressed in units of energy (u.e.) and the maximum charging/discharging powers \(C^\text{max}\) and \(D^\text{max}\) are expressed in units of power (u.p.). For a variability \(\sigma\) and a ramping capability \(\zeta\), we choose that one unit of energy is equal to \(\sigma^2/\zeta\) and one unit of power corresponds to \(\sigma^2/\zeta^2\). We refer to Section 7 for the reasons for these scalings.

For example, let us consider the scenario envisioned for the UK in 2020 [3, 11] where wind power is used to cover 20% of the total electricity consumption – this corresponds to 26GW of peak power. At the scale of the country, the variability of the demand is small compared to the variability of the wind generation. This variability is due to uncertain
forecast. Using the same data as [11], the square of the volatility of the wind generation is \( \sigma^2 = 0.6 \) (GW)/h. This means that if the ramping capability \( \zeta = 1 \) GW/h, one unit of energy corresponds to 360MWh and one unit of power to 600MW. For a ramping capability of \( \zeta = 2 \) GW/h, one unit of energy corresponds to 45MWh and one unit of power to 300MW.

Throughout the paper, all numerical values are obtained by discretizing in time and space the original model to obtain a discrete Markov decision process. This control problem is then solved using brute-force dynamic programming to obtain the function \( b \mapsto \phi(b) \). The simulation of a sample path (Figures 4(b) and 5) is obtained using \( \phi \) in a discrete-time continuous-space model and the steady-state performance indicator (Figures 6 to 11(b)) are computed by computing the stationary measure of the discrete-time discrete-space model. Unless otherwise specified, the parameters used for the simulations are \( \sigma=1, \zeta^+=1, \zeta^- = 3, B_{\text{max}} = 1 \) u.e., \( C_{\text{max}}=D_{\text{max}}=3 \) u.p., \( \eta=1, \gamma=0.01, c=1 \) and \( (v + c^{bo})=5 \).

5. DYNAMIC COMPETITIVE EQUILIBRIA AND MARKET EFFICIENCY

Under a dynamic competitive equilibrium, the price process is such that the decisions taken over time by the various market actors maximize their respective welfare. In this section, we first prove that for all scenarios, the market is efficient in the sense that any dynamic competitive equilibrium maximizes the social welfare. We then prove the existence of such an equilibrium: we provide explicit expressions for the equilibrium price process and for the strategic controls used by the various market actors in equilibrium. We conclude the section by showing numerically the effect on prices of the presence of storage: in absence of storage, prices are volatile and can only take two values, 0 and the "choke-up" price \( v + c^{bo} \) [8], whereas with storage, prices are smoother and oscillate around the marginal production cost \( c \) as the storage energy capacity grows large.

5.1 Market Efficiency

We first assume that dynamic competitive equilibria exist, and we prove that these equilibria are efficient in the sense that the corresponding controls maximize the system social welfare. This result is often referred to as the social welfare theorem in economics. A similar result has been established in [26] without storage. We show that the market remains efficient even in presence of storage.

**Theorem 2. (Social Welfare Theorem)** Assume that a dynamic competitive equilibrium exists. Then:

(i) any competitive equilibrium maximizes the social welfare;

(ii) conversely, for any control processes \((E_0^*, E_1^*, G^*, u^*)\) maximizing the social welfare, there exists a price process \(P^*\) such that \((P^*, E_0^*, E_1^*, G^*, u^*)\) is a competitive equilibrium.

5.2 Equilibria: Existence and Properties

If dynamic competitive equilibria exist, we know that they are socially efficient (even in presence of storage). We show that competitive equilibria indeed exist and characterize the corresponding price and control processes. More precisely, we identify a price process \(P^*\) that can lead to an equilibrium (in fact, as it turns out, \(P^*\) is the only price process leading to an equilibrium), and show that if \((E_0^*, E_1^*, G^*, u^*)\) are socially optimal controls, \((P^*, E_0^*, E_1^*, G^*, u^*)\) is a competitive equilibrium.

Let \((R^*, u^*)\) be the reserve-storage control process maximizing the social welfare (starting at \(R^*(0) = r\) and \(B^*(0) = b\)). Denote by \(B^*\) the corresponding storage level process, and define the price process \(P^*\) as:

\[
P^*(t) = \left\{ \begin{array}{ll}
0 & \text{if } R^*(t)+u^*(t) > 0, \\
\eta \frac{\partial V}{\partial b}(R^*(t), B^*(t)), & \text{if } R^*(t)+u^*(t) = 0, R^*(t) > 0, \\
\frac{\partial V}{\partial b}(R^*(t), B^*(t)), & \text{if } R^*(t)+u^*(t) = 0, R^*(t) \leq 0, \\
v + c^{bo}, & \text{if } R^*(t)+u^*(t) < 0.
\end{array} \right.
\]

**Theorem 3.** Let \((P^*, G^*, u^*)\) be defined as above. Then

(i) \((P^*, E_0^* = \Gamma + G^* + u^*, E_1^* = \Gamma + G^* + u^*, G^*, u^*)\) is a competitive equilibrium when the storage is at the supplier.

(ii) \((P^*, E_0^* = \Gamma + G^*, E_1^* = \Gamma + G^* + u^*, G^*, u^*)\) is a competitive equilibrium when the storage is at the consumer.

(iii) \((P^*, E_0^* = \Gamma + G^* + u^*, E_1^* = \Gamma + G^* + u^*, G^*, u^*)\) is a competitive equilibrium when there is a stand-alone storage owner.

In particular, this theorem implies that at the equilibrium, the price, generation and storage control processes are the same in all three scenarios. Moreover these controls maximize the social welfare.

5.3 Equilibrium Price Distribution: The Impact of Storage

At the equilibrium, the price process \((11)\) is a function of the optimal reserve and storage control processes. We plot in Figure 5 the evolution over time of the prices \(P^*(t)\) and of the storage level \(B^*(t)\). We fix the maximum charging and discharging powers to \(C_{\text{max}}=D_{\text{max}}=3\) u.p. and we compare the results for four values of energy capacity of storage: \(B_{\text{max}}=0\) u.e. (i.e. no storage), 1 u.e., 3 u.e. and 10 u.e., where 1 u.p. = \(\sigma^2/\zeta^+\) and 1 u.e. = \(\sigma^2/\zeta^-\) (see §4.3). We use the same random seed so that \(D(t)-\Gamma(t)\) is the same in all four cases.

As shown in [26], when there is no storage, the prices oscillate between 0 and the "choke-up" price \((v + c^{bo})\) (Figure 5(a)). When \(B_{\text{max}} > 0\), the prices always remain between 0 and \((v + c^{bo})\), but they are smoother. When \(B_{\text{max}}\) is large and \(\eta = 1\) (Figure 5(c)), the prices are almost constant and close to the marginal cost of production \(c = 1\). When \(B_{\text{max}}\) is large and \(\eta = 0.8\) (Figure 5(d)), the prices oscillate around \(c = 1\), in this case, between 0.88 and 1.1 = 0.88/\(p\).

The optimal reserve and storage level process \((R^*, B^*)\) is stationary. We compute numerically the distribution of
prices in the steady-state regime of the process \((R^*, B^*)\). The results are reported in Figure 6. Again, we observe that when there is no storage, the price takes only the two values \(0\) and \((v + c^{bo})\). When \(\eta = 1\), the price tends to concentrate around the marginal production cost \(c = 1\) as \(B_{\text{max}}\) increases. For \(\eta = 0.8\), the price signals whether the storage owner should charge (price\(\approx0.88\)) or discharge (price\(\approx1.1\)).

6. STRATEGIC INVESTMENT IN STORAGE

In this section, we study numerically the welfare of the different actors as a function of the energy capacity of storage and of the maximum charging/discharging powers. We first define and compute the socially optimal energy capacity (§6.1). We then show that when the supplier owns the storage, her optimal energy capacity is the socially optimal one. However, when the storage controlled by the consumer or by an independent actor, their optimal energy capacity is strictly lower. Finally, we show in §6.3 that the same results hold for the maximum charging/discharging powers.

6.1 Socially Optimal Energy Capacity

The total payoff, given by (8), is composed of two terms: the benefit for the consumer minus the cost of black out, equal to \(-cG(t)\), and minus the cost of producing the energy \(-cG(t)\). Thus, if the initial reserve and storage process \((r, b)\) is distributed as the stationary distribution \(\pi\) of the optimal reserve and storage processes, the expected social welfare is

\[
-\frac{1}{\gamma}E_{(r,b)\sim \pi} \left( \int_0^{\infty} e^{-\gamma t} \left( (v + c^{bo})(R^* + u^*) - cR^* \right) dt \right)
\]

The social welfare increases as the energy capacity \((B_{\text{max}})\) or maximum charging/discharging power \((C_{\text{max}}\) and \(D_{\text{max}})\) of the storage system increases. Thus, if we neglect the cost of installing additional energy capacity, the greater the storage system is, the greater the social welfare is.

![Figure 5: Evolution of prices and storage level over time for various storage energy capacities \(B_{\text{max}}\). For \(B_{\text{max}} = 10\), we compare \(\eta = 0.8\) and \(\eta = 1\).]

![Figure 6: Steady-state distribution of prices for various storage energy capacities \(B_{\text{max}}\). For \(B_{\text{max}} = 10\), we zoom on \(c=1\) to compare \(\eta = 0.8\) and \(\eta = 1\).]

![Figure 7: Expected social welfare in the stationary regime as a function of the energy capacity \(B_{\text{max}}\).]
0 when $R^*(t) + u^*(t) > 0$ and to $(v + c^{bo})$ when $R^*(t) + u^*(t) < 0$. This shows that the supplier’s payoff equals $-(v + c^{bo})(R^*(t) + u^*(t)) - cR^*(t)$, which is exactly the total payoff. This shows that if the storage belongs to the supplier, her welfare increases as the energy capacity increases in the same proportion as the social welfare. Hence, the optimal energy capacity from a supplier-owned storage perspective is the same as the socially optimal capacity.

### 6.2.2 Storage at the Consumer and at a Stand-Alone Storage Operator

When the consumer owns the storage, her payoff at time $t$ is equal to $-(v + c^{bo})(R^*(t) + u^*(t)) - P^*(t)R^*(t)$. The price $P^*(t)$ is equal to 0 when $R^*(t) + u^*(t) > 0$ and to $v + c^{bo}$ when $R^*(t) + u^*(t) < 0$. Thus we can write

$$P^*(t)R^*(t) = P^*(t)(R^*(t) + u^*(t)) - P^*(t)u^*(t) = -(v + c^{bo})(R^*(t) + u^*(t)) - P^*(t)u^*(t).$$

This shows that the consumer’s payoff is equal to $P^*(t)u^*(t)$. If the storage is owned by a stand-alone storage operator, her instantaneous payoff will be exactly the same: at time $t$, a quantity $u^*(t)$ is sold at price $P^*(t)$. Thus the expected welfare of the consumer is equal to that of a stand-alone storage operator:

$$E_{t,h} = \frac{1}{\gamma} E_{t,h} \left[ P^*(t)u^*(t) \right].$$

In Figure 8, we plot the average welfare of a stand-alone storage operator as a function of the energy capacity $B_{max}$. The system parameters are the same as those used to compute the social welfare in Figure 7. When $C_{max} = D_{max} = 1$, the expected welfare saturates for $B_{max} = 2$ u.e. and diminishes slightly afterward. When $C_{max} = D_{max} = 3$ u.p., the expected welfare is maximal for $B_{max} \approx 1.5$ u.e. and decreases sharply afterward. It diminishes almost to zero for $\eta = 0.8$ when $B_{max}$ goes to infinity. In both cases, the expected welfare is maximal for a finite energy capacity and this capacity is much lower than the socially optimal capacity. This means that the consumers and the stand-alone storage operators have an incentive to undersize their storage.

This result seems paradoxical, as it implies that even if we neglect the cost of storage, a storage owner would make less money with a larger energy capacity. The explanation comes from the price-taking assumption (A2). When the energy capacity grows, the price variations diminish. As a stand-alone storage owner gains only from buying at low price and selling at high price, her gain diminishes as the prices variability decreases. This situation is radically different when the supplier owns the storage. In this case, a higher storage leads to lower losses and therefore diminishes the production cost. This results in higher gain for the supplier even with a large storage.

### 6.3 Optimal Maximum Charging/Discharging Powers

As we just observed, when the storage belongs to the consumer or to a third party, the storage owner has an incentive to undersize her energy capacity, compared to a social planner. Figure 9 shows that we have a similar phenomenon regarding the maximum charging/discharging power.

In Figure 9(a), we plot the expected social welfare as a function of the maximum charging/discharging power $C_{max} = D_{max}$. This curve is similar to Figure 7 and the optimal charging/discharging powers are $C_{max} = D_{max} \approx 1$ u.p.. In Figure 9(b), we plot the expected welfare of a stand-alone storage operator. As for the energy capacity, it has a maximum for a relatively small value $C_{max} = D_{max} \approx 1$ u.p. and then decreases quickly. We plot these curves for $B_{max} = 5$ u.e. but their shapes are similar for other values of $B_{max}$, even small values like $B_{max} = 0.5$ u.e..

**Figure 9:** Welfare of the players as a function of maximum charging/discharging power $C_{max} = D_{max}$.

### 7. VOLATILITY AND STORAGE SIZING

We now study the effect of $\sigma$ (the volatility of the demand and renewable generation process) on the way the storage should be dimensioned. We show that if the ramping capability of the generators remains unchanged, then the storage energy capacity needed to mitigate this volatility scales as $\sigma^4$, and that the required charging/discharging powers scale as $\sigma^2$. If the volatility is fixed and the ramping capability $\zeta$ varies, then the storage capacity needed to mitigate the volatility scales as $1/\zeta^3$, and the maximum charging/discharging powers should scale as $1/\zeta$. In particular, this implies that to maintain a fixed optimal energy capacity, $\zeta$ should scale as $\sigma^{4/3}$.

This supports the idea that increasing the amount of power coming from renewables could lead to an increase in carbon dioxide emissions. Doubling the amount of renewables would result in multiplying the volatility by 2. In this case, it is probably easier and less expensive to install generators with $2^{4/3} \approx 2.5$ times larger ramping capability rather than to multiply the energy capacity by 16. However, these fast-ramping generators (e.g. gas turbines) tend to emit more carbon dioxide than conventional generators.
7.1 Scaling Laws for the Storage Capacities

Let us recall that $\sigma$ is the volatility of the difference of demand and of the renewable generation process and that $\zeta^-$ and $\zeta^+$ are the ramping capabilities of the real-time generators. The next theorem shows that if the storage capacity $B_{\text{max}}$ is scaled as $\sigma^4/\zeta^3$ and the maximum charging/discharging powers $C_{\text{max}}$ and $D_{\text{max}}$ are scaled as $\sigma^2/\zeta$, then the social welfare scales as $\sigma^2/\zeta$.

In Section 6, we defined the socially optimal storage parameters. They correspond to the knee of the curve of Figures 7 and 9(a) beyond which installing new storage capacity leads to a negligible increase of the social welfare. Therefore, this theorem implies that the optimal storage energy capacity needed to accommodate the volatility scales as $\sigma^4/\zeta^3$, and the maximum charging/discharging powers scale as $\sigma^2/\zeta$:

$$B_{\text{max}} = \Theta(\sigma^4/\zeta^3) \quad \text{and} \quad C_{\text{max}} = \Theta(\sigma^2/\zeta).$$

**Theorem 4.** Let $R, B$ be the socially optimal reserve and storage level processes when the system parameters are $(\sigma, \zeta^+, \zeta^-, B_{\text{max}}, C_{\text{max}}, D_{\text{max}}, \gamma)$.

Then, when the system parameters become

$$(x\sigma, y\zeta^+, y\zeta^-, x^2/2B_{\text{max}}, x^2/yC_{\text{max}}, x^2/yD_{\text{max}}, x^2/y^2),$$

the corresponding socially optimal processes $R^{x,y}, B^{x,y}$ are

$$R^{x,y}(t) = x^2/y B(y^2/t) \quad \text{and} \quad B^{x,y}(t) = x^2/y B(y^2/t).$$

Moreover, let $W$ be the social welfare for the initial parameters. The social welfare for the rescaled parameters is $(x^2/y)W$.

The proof of this theorem consists in verifying that the proposed scaling works. It is detailed in [10].

7.2 Practical Implications

Theorem 4 implies that if $\zeta$ and $\sigma$ are multiplied by the same factor $x$, then the optimal storage parameters increase linearly in $x$. Figure 10 illustrates this fact: we plot the optimal social welfare as a function of the available $B_{\text{max}}$ for four values of $x$. Each time the remaining parameters are the same, and we choose $C_{\text{max}} = D_{\text{max}} = 1000$ to release the charge/discharging constraints of the storage. Figure 10 shows that for $x = 1$ (respectively 2, 4, 6), the optimal energy capacity is approximately 8 (respectively 15, 30 and 50), that is to say linear in $x$ (as expected).

For a fixed ramping constraint $\zeta$, Theorem 4 states that the optimal energy capacity increases as the scaling factor of $\sigma$ to the fourth power. In Figure 11(a) we scale $\zeta$ by the a fixed constant $y = 3$, we scale $\sigma$ by a various factors $x = 1, 2, 3, 4$, and we plot the resulting optimal social welfare. For the four values of $x$, the respective optimal values of the energy capacity are approximately 7, 100, 400, 1000 u.e., which correspond to roughly $\Theta(x^4)$, as expected.

For a fixed scaling of the volatility $\sigma$, by Theorem 4 the optimal storage size decreases as the cube of the scaling factor of the ramping constraint $\zeta$. This is illustrated in Figure 11(b), where we consider a fixed $\sigma$, and we scale $\zeta$ by factors $y = 1, 2, 3, 4$. We plot the optimal social welfare, and we observe the decreasing values of optimal energy capacity 10, 1.5, 0.5, 0.3 ($\approx \Theta(y^{-3})$).

8. CONCLUSION

We have shown that under the price-taking assumption, electricity markets remain efficient when we introduce storage capabilities in the system: they lead to an efficient allocation of generation and storage control and smooth the prices. However, we have shown that there is no incentive for consumers or third-party actors to install large storage devices, despite the fact that the required storage capacity needed to accommodate real-time fluctuations explodes with high variability.

There are still many open questions. A first question concerns the case of oligopolies: Does the market remain efficient if a small number of players can influence prices? Another question is whether the market efficiency results also hold in a physical network as described in [26]. More specifically, how are these results affected by the placement of storage devices in such a network? Finally, the lack of incentives for actors to install large storage capacity raises the question of designing political incentives to encourage the development of storage systems.
APPENDIX

A. PROOF OF THEOREM 1

To derive socially optimal controls, we use the following structural properties for the value function $V$:

**Lemma 1.** (i) $(r, b) \to V(r, b)$ is concave.
(ii) $V$ is sub-additive: for all $r^- \leq r^+ \leq b^-$ and $b^- \leq b^+$,

$$V(r^-, b^-) + V(r^+, b^+) \leq V(r^-, b^+) + V(r^+, b^-).$$

(iii) For all $b^- \leq b^+$, and $r$:

$$0 \leq V(r, b^+) - V(r, b^-) \leq (b^+ - b^-)(v + c_{bo}).$$

The proof of the lemma is presented at the end of this section. We first prove the optimality of $g^*$. We start from a reserve $R(0) = r$ and storage $B(0) = b$. The storage control is fixed and optimal. Due to the concavity of $V$, $\frac{\partial V}{\partial r}$ is well defined almost everywhere. Let $\delta > 0$. We denote by $d(\delta) = \int_0^\delta Z(t)dt$, and by $b' = B(\delta)$. Consider two generation controls $g_1$ and $g_2$ different over $[0, \delta]$, but equal to the optimal control after $\delta$. Denote by $V_1$ and $V_2$ the expected social welfare obtained using $g_1$ and $g_2$, respectively. Define $\xi_1 = \int_0^\delta g_1(t)dt$ and $\xi_2 = \int_0^\delta g_2(t)dt$. We can show that:

$$V_1 - V_2 = e^{-\gamma\delta}\mathbb{E}[V(r + \xi_1 - d(\delta), b')] - V(r + \xi_2 - d(\delta), b') + O(\delta^2)$$

$$= \delta(\xi_1 - \xi_2)\frac{\partial V}{\partial r}(r, b) + O(\delta^2).$$

We deduce that the optimal generation control $g^*$ is such that $g^* = \xi^*$ if $\frac{\partial V}{\partial r} > 0$ and $g^* = \xi^*$ if $\frac{\partial V}{\partial r} < 0$. Now since $V$ is concave, the threshold function $\phi(b) = \sup\{r : \frac{\partial V}{\partial r} \geq 0\}$ is well defined, and for $r < \phi(b)$ (resp. $r > \phi(b)$), the optimal generation control is $g^* = \xi^+$ (resp. $\xi^-$). The fact that $\phi$ is nonincreasing is directly deduced from (ii) in Lemma 1.

Next we prove the optimality of $u^*$. We establish that $u^*$ is optimal when starting from a reserve $r = R(0)$ over a small time interval $[0, \delta]$. The generation control is fixed, and we denote by $r' = R(\delta)$. Let $b = B(0)$. The proof consists in analyzing several cases depending on the initial reserve value $r$. Due to space limitation, we provide the analysis of one of the cases. The other cases are treated in [10].

Assume that $r \in (0, C_{max})$ and $\delta > 0$ is such that $C_{max} \geq R(\delta) \geq 0$ for all $s \in [0, \delta]$ (with high probability). We have $u^*(s) = -R(s)$ for all $s \in [0, \delta]$. If this control is used, the expected social welfare is denoted by $V_1$. Now assume that instead we use the control $u \in [-C_{max}, C_{max}]$, in which case, the expected welfare is $V_2$. At time $t$, under control $u^*$, the state of the storage is $b + c_1$, and under $u$, it is $b + c_2$. We have $c_1 = \int_0^\delta \eta R(s)ds$, and $c_2 = -\int_0^\delta \eta(u(s)1_{u(s) > 0} + u(s)1_{u(s) < 0})ds$

$$\leq -\int_0^\delta \eta(u(s)1_{u(s) < -R(s)} + u(s)1_{-R(s) < u(s) < 0})ds$$

$$\leq -\int_0^\delta \eta(u(s)1_{u(s) < -R(s)} - R(s)1_{-R(s) < u(s) < 0})ds.$$}

We deduce that: $c_2 - c_1 \leq \int_0^\delta (R(s) + u(s))^+ds$. Now observe that under control $u$, if $R(s) + u(s) < 0$ we have a blackout, and hence:

$$V_1 - V_2 = (v + c_{bo})\int_0^\delta dse^{-\gamma s}(R(s) + u(s))^- + e^{-\gamma\delta}\mathbb{E}[V(r', b + c_1) - V(r', b + c_2)].$$

From (iii) in Lemma 1,

$$V_1 - V_2 \geq (1 - \eta)(v + c_{bo})\int_0^\delta ds(R(s) + u(s))^+ \geq 0.$$}

**Proof of Lemma 1.** (i) and (ii) are straightforward and details can be found in [10]. Regarding (iii), let $b^- \leq b^+$ and $u_-$ and $u_+$ be optimal controls starting from $b^-$ and $b^+$. Let $B_s$ be the storage level process starting at $b^+$ under control $u_+$. We introduce the control $u$ and the corresponding storage level process $B$ that starts from $b^-$:

$$u(t) = \begin{cases} u_+(t) & \text{if } B(t) > 0 \text{ or } u_+(t) < 0 \\ 0 & \text{otherwise} \end{cases}$$

By abuse of notation, we denote $V(u)$ the value function of the control $u$. As $u_-$ is optimal starting from $b^-$ and $u$ is a valid control starting from $b^+$, we have:

$$V(b^+) - V(b^-) = V(u_+) - V(u) = E[\int_0^\infty e^{-\theta(t)}(v + c_{bo})(R(t) + u(t))^+ - (R(t) + u_+(t))^+dt]$$

$$\leq E[\int_0^\infty (v + c_{bo})(u_+(t) - u(t))dt] \leq (v + c_{bo})(b^+ - b^-).$$

The first inequality is obtained remarking that $x \mapsto -(R + x)^+$ is 1-Lipschitz. The last inequality is obtained by combining the facts that $b^+ - B_+(T) = \int_0^T f(u_+(t))dt$, $b^- - B(T) = \int_0^T f(u(t))dt$, where $f(x) = x(1_{x > 0} + \eta1_{x \leq 0})$, and that by construction $B(T) \leq B_+(T)$ for any $T$, the inequality is deduced by letting $T \to \infty$.

B. PROOF OF THEOREM 2

To simplify notations, we define $E = (E_D, E_S, G, u)$ and we write $E \in X_C$ if $E_D \in X_D$ and $(E_S, G, u) \in X_S$. To emphasize the dependence of the social welfare on the process $E$, we use the notations $W(E)$. Similarly we denote by $W_D(E_D, P)$ and $W_S(E_S, G, u, P)$ the welfare of the consumer and the supplier. Moreover, we define the inner product of two stochastic processes $F_1$ and $F_2$ as: $\langle F_1, F_2 \rangle := E\int_0^\infty e^{-\gamma t}(F_1(t)F_2(t))dt$. Using this notation, the consumer’s welfare becomes $W_D := \langle W_D, 1 \rangle$, where $1$ is the constant process, always equal to 1. The proof is detailed for scenario A. It applies mutatis mutandis to the two other scenarios.

To prove the result, we proceed as in [26] and we interpret the social welfare optimization problem as the problem of maximizing $W_D(E_D, P) + W_S(E_S, G, u)$ subject to the constraints $E \in X_C$ and $E_D = E_S$. The last constraint is relaxed, and the price process $P$ serves as corresponding Lagrange multipliers. The Lagrangian is:

$$\mathcal{L}(E, P) = -W(E) + \langle P, E_D - E_S \rangle$$

$$= -W_D(E_D, P) - W_S(E_D, G, u, P).$$

The dual function $h$ is defined as: $h(P) = \inf_{E \in X_C} \mathcal{L}(E, P)$. Let $E \in X_C$. If $E_D = E_S$, weak duality holds: $h(P) \leq \mathcal{L}(E, P) = -W(E)$. We further establish that for $E^* \in X_C$, such that $E^*_D = E^*_S$, $(E^*, P)$ is a competitive equilibrium if
and only if \( h(P) = -\mathcal{W}(E) \).
1. Assume that \((E^*, P)\) is a competitive equilibrium. Then:

\[
h(P) = \inf_{k \in X_C} \mathcal{L}(E, P) = -\sup_{E \in X_D} \mathcal{W}_D(E_D, P) - \sup_{(E, G, u) \in X_S} \mathcal{W}_S(E_S, G, u, P) = -\mathcal{W}_D(E^D, P) - \mathcal{W}_S(E^S, G^*, u^*, P) = \mathcal{L}(E^*, P) = -\mathcal{W}(E).
\]

2. Conversely, assume that under \((E^*, P)\), \( h(P) = -\mathcal{W}(E) \). Since \((E_D^*, E_S^*) \in X_T\), we deduce that: \( h(P) = \mathcal{L}(E^*, P) \), which in turn implies that:

\[
-\sup_{E_D \in X_D} \mathcal{W}_D(E_D, P) - \sup_{(E, G, u) \in X_S} \mathcal{W}_S(E_S, G, u, P)
= -\mathcal{W}_D(E_D^*, P) - \mathcal{W}_S(E^S, G^*, u^*, P).
\]

We conclude that \(E^*_D\) maximizes the welfare from the consumer’s perspective, and that \((E^S, G^*, u^*)\) maximizes supplier’s welfare under price process \(P\).

From the above analysis, we deduce that any competitive equilibrium maximizes the social welfare. Now let \((E^*, P^*)\) be a competitive equilibrium, and let \(E \in X_C\) be socially optimal. Since both \(E\) and \(E^*\) maximize the social welfare, we have: \(-\mathcal{W}(E) = -\mathcal{W}(E^*) = h(P^*)\). This implies that \((E, P^*)\) is a competitive equilibrium.

### C. PROOF OF THEOREM 3

The proof is detailed for scenario A. The two other scenarios B and C can be treated analogously.

The proof of the theorem consists in showing that under the price process \(P^*\), the controls maximizing the social welfare are also optimal from the consumer’s and supplier’s perspectives. We first prove the following result.

**Lemma 2.** We have: \( \frac{\partial}{\partial r} = \mathbb{E} \int_0^\infty e^{-\gamma t}(P^*(t) - c)dt \).

**Proof.** Let \( \epsilon > 0 \) (typically small). We first prove that we can choose \( \delta > 0 \) uniformly w.r.t. \((r, b)\) such that:

\[
V(r, b) - V(r + \epsilon, b) \geq \epsilon \mathbb{E} \int_0^\infty e^{-\gamma t}(-1 + \epsilon)P^*(t) + c)dt
+ e^{-\gamma \delta}\mathbb{E}[V(R^*(\delta), B^*(\delta)) - V(R^*(\delta) + \epsilon, B^*(\delta))]. \tag{14}
\]

Let \((R_1, B_1) \) (resp. \((R_2, B_2)\)) be the socially optimal reserve and storage level processes when starting at \((r, b)\) (resp. \((r + \epsilon, b)\)). Observe that the controls leading to the processes \((R_1 + \epsilon, B_1)\) are suboptimal when starting from \((r + \epsilon, b)\). We deduce that: \( V(r, b) - V(r + \epsilon, b) \geq G(\delta) + H(\delta) \), where

\[
G(\delta) = \mathbb{E} \int_0^\delta e^{-\gamma t}(W(R_1(t), B_1(t)) - W(R_2(t), B_2(t)))dt,
H(\delta) = e^{-\gamma \delta}\mathbb{E}[V(R_1(\delta), B_1(\delta)) - V(R_1(\delta) + \epsilon, B_2(\delta))].
\]

We consider several cases depending on the initial reserve value \(r\). Due to space limitations, we provide the analysis of one of the cases — the other cases are treated in details in [10].

Assume that \( 0 < r < C_{\text{max}} \) and for all \( t \leq \delta \), \( R_1(t), R_2(t) \in [0, C_{\text{max}}] \). In this case, for all \( t \leq \delta \), \( u_1(t) = -R_1(t) \) and \( u_2(t) = -R_2(t) \). Note that \( R_1(t) + \epsilon \) is suboptimal when starting from \((r + \epsilon, b)\) and so we can assume that \( u_2(t) = -R_1(t) - \epsilon \). This leads to \( B_2(\delta) = B_1(\delta) + \epsilon \).

We have: \( G(\delta) \geq \epsilon \mathbb{E} \int_0^\delta e^{-\gamma t}dt \), and

\[
H(\delta) = e^{-\gamma \delta}\left(\mathbb{E}[V(R_1(\delta), B_1(\delta)) - V(R_1(\delta) + \epsilon, B_2(\delta))] + \mathbb{E}[V(R_1(\delta) + \epsilon, B_1(\delta)) - V(R_1(\delta) + \epsilon, B_1(\delta) + \epsilon \delta \eta)]\right).
\]

We conclude that:

\[
\mathbb{E}[-\mathcal{W}(E^*, P^*) + \mathcal{W}(E, P)] \geq -\epsilon(1 + \epsilon)\mathbb{E} \int_0^\infty e^{-\gamma t}dt.
\]

We deduce that:

\[
G(\delta) + H(\delta) \geq \epsilon \mathbb{E} \int_0^\delta e^{-\gamma t}(-1 + \epsilon)P^*(t) + c)dt
+ e^{-\gamma \delta}\mathbb{E}[V(R_1(\delta), B_1(\delta)) - V(R_1(\delta) + \epsilon, B_1(\delta))].
\]

Using a similar analysis for the other cases (see [10]), Equation (14) follows. Iterating (14), we obtain that for any \( k > 0 \):

\[
V(r, b) - V(r + \epsilon, b) \geq \epsilon \mathbb{E} \int_0^\delta e^{-\gamma t}(-1 + \epsilon)P^*(t) + c)dt
+ e^{-\gamma \delta}\mathbb{E}[V(R^*(\delta), B^*(\delta)) - V(R^*(\delta) + \epsilon, B^*(\delta))].
\]

Letting \( k \to \infty \), and then \( \epsilon \to 0 \), we conclude that:

\[
\mathbb{E} \int_0^\infty e^{-\gamma t}(P^*(t) - c)dt.
\]

The converse inequality is obtained by similar arguments.

We now prove that \((E_D^*, \Gamma + G^* + u^*)\) optimizes the consumer’s welfare under price \(P^*\). The consumer’s welfare is:

\[
\mathbb{E} \int_0^\infty e^{-\gamma t}(-v + e^\mu_0)(E_D(t) - D(t) + rda) - P^*(t)E_D(t)dt.
\]

As there is no ramping constraints in the consumer’s optimization (i.e., \(E_D \in X_D\)), an optimal strategy is myopic in the sense that for any \( t > 0 \), an optimal strategy \( E_D(t) \) satisfies \( E_D(t) \in \text{arg max}_{(E_D, E_S)} \{-v + e^\mu_0(e^{-D(t) + r*a} - P^*(t))\} \).

Recall that \( R^* = G^* + \Gamma + D + rda \). We consider three cases:

1. When \( P(t) = 0 \) (i.e. \( R^*(t) + u^*(t) > 0 \)) any \( \epsilon \geq D(t) - rda \) maximizes the consumer’s payoff. Thus \( E_D(t) = \Gamma(t) + G^*(t) + u^*(t) \).

2. When \( \epsilon < D(t) \) (i.e. \( R^*(t) + u^*(t) < 0 \)) her payoff is maximized for \( E_D(t) = D(t) - rda \).

3. When \( P(t) = v + e^\mu_0 \), her payoff is maximized for any \( E_D^*(t) \leq D(t) + rda \). Thus \( E_D(t) = \Gamma(t) + G^*(t) + u^*(t) \).

We conclude that \((R^*, u^*)\) maximizes the welfare of the supplier. By assumption (A1-A3), the real time price \(P(t)\) and the generation of renewable energy \(\Gamma(t)\) are uncontrollable by the players. Moreover, as \( E_S(t) \leq \Gamma(t) + u(t) + G(t) \) and the payoffs of the supplier is increasing in \( E_S \), a supplier will chose \( E_S(t) = \Gamma(t) + u(t) + G(t) \). Thus, a direct computation shows that the welfare of the supplier can be solely expressed as a function of the reserve process \(R \).

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and the control process $u$. Her payoff can be written (up to uncontrollable parts) as $W_S(t) = P^r(t)(R(t) + u(t)) - cR(t)$.

We define by $X_R$ the set of possible generation controls: $(R, u) \in X_R$ if $R \in X_R$ and $u \in X_R$. This decomposition is possible because the constraints on generation and storage are not coupled. Observe that the optimal control problem that the supplier solves is equivalent to:

$$\max_{(R,u) \in X_R} \mathbb{E} \int_0^\infty e^{-\gamma t}(P^r(t)(R(t) + u(t)) - cR(t))dt.$$

The corresponding value function $V_S(t', b)$ is:

$$V_S(t', b) = \sup_{(R,u) \in X_S} \mathbb{E}_{(t,b)} \int_0^\infty e^{-\gamma t}(P^r(t)(R(t) + u(t)) - cR(t))dt.$$

We have $V_S(t', b) = V_R(t') + V_B(b)$ where

$$V_R(t') = \sup_{R \in X_R} \mathbb{E}_{t'} \int_0^\infty e^{-\gamma t}(P^r(t) - c)R(t)dt$$

and

$$V_B(b) = \sup_{u \in X_R} \mathbb{E}_b \int_0^\infty e^{-\gamma t}P^r(t)u(t)dt.$$

Optimality of $R^r$: From the above expression, we simply deduce that: $\frac{\partial R^r}{\partial b} = \mathbb{E}_{t'} \int_0^\infty e^{-\gamma t}(P^r(t) - c)dt$. From Lemma 2, we conclude that $\frac{\partial V_S}{\partial b} = \frac{\partial V}{\partial b}$ and hence $R^r$ also maximizes the supplier’s welfare.

Optimality of $u^*$: The proof is similar to that of the social optimality of $u^*$. It relies on the observation that $V_B(b') - V_B(b) \leq (v + \kappa b)\Delta b$. See [10] for details.

D. REFERENCES


