Nontransitive Dice Paradox in Networking *

[Poster Abstract]

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Here is a simple trick that you can use to try to dupe a friend out of his money with high probability. Bet with him on the outcome of three specially numbered dice labelled X, Y, and Z each with six faces marked as: X (3 3 5 5 7 7), Y (2 2 4 4 9 9), Z (1 1 6 6 8 8). Each person chooses a different die. The larger number wins when the dice are rolled. Strangely, whichever die your friend has chosen, you can always outwit him with another die of greater winning probability; in fact $\mathbb{P}(X > Y) > \mathbb{P}(Y > X)$, $\mathbb{P}(Y > Z) > \mathbb{P}(Z > Y)$, and $\mathbb{P}(Z > X) > \mathbb{P}(X > Z)$. Therefore, one can legitimately arrange the order of the dice as $Z \succ X \succ Y \succ Z$, creating a cyclic preference.

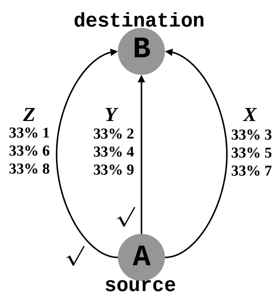
This is an amusing paradox that also arises in decision problems and elections, where the relation induced by pair-wise comparisons on random variables, unlike on deterministic values, may violate transitivity. Indeed, there is an implication in networking, and this poster shows that the behaviour of distributed decision making in the presence of stochastic metrics gives rise to situations resembling the nontransitive dice game with counter-intuitive outcomes that may trap unwary system designers and network administrators.

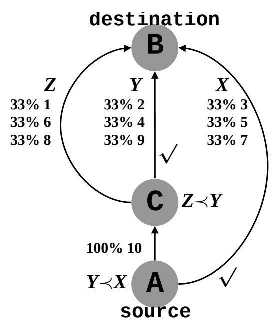
Networks are plagued with uncertain metrics. Internet traffic is inherently stochastic due to both fluctuating demands and varying payload sizes, leading to stochastic metrics such as available bandwidth, queuing delay, throughput, and jitter. Link states are not always static, for instance, due to changes of network topology caused by system failures and misconfigurations. Furthermore, connections over novel network infrastructures, such as overlay networks and wireless ad hoc networks, give rise to more varying and dynamic connectivity. Often, routers will be supplied with incomplete or out-dated knowledge of the network, owing to delayed, infrequent or batched link state advertisements. In addition, aggregation of link states in hierarchical networks, in order to help scalability, but presents imprecise information to identify the available resources.

It is feasible to acquire the information about path characteristics such as available bandwidth locally by using a variety of end-to-end measurement tools, based on active probing or passive inference. For example, in overlay networks such as RON or Planetlab, information about the available bandwidth of the paths is useful for making local routing decision. Given that the metrics for deciding the best path are stochastic, distributed decision making is not as simple as in the deterministic case.

Consider an overlay network, with three paths X, Y, Z with available bandwidth distributed as the nontransitive dice. Drawing on the nontransitive dice game, one can see that different arrangements of comparisons will yield different paths of maximum bandwidth. If all three paths are put together as in the left diagram below, then Y and Z are the optimal paths with the most likely maximum bandwidth since $\mathbb{P}(\max\{X,Y,Z\} = Y) = \mathbb{P}(\max\{X,Y,Z\} = Z) > \mathbb{P}(\max\{X,Y,Z\} = X)$.

However, with a different topology as in the right diagram below, contrary to A's intent, it has to choose the *least* likely maximum bandwidth path X. It is because the intermediate node C selects path Y as it is stochastically larger than Z (i.e. $\mathbb{P}(Y > Z) > \mathbb{P}(Z > Y)$). Hence, A is forced to choose path X because $\mathbb{P}(X > Y) > \mathbb{P}(Y > X)$. Here arises the paradox.





Motivated by the above paradox, we explore the consequences of network algorithms in the presence of stochastic metrics. We identify several more paradoxical cases, including a Braess-like paradox where the addition of link capacity by a well-meaning network administrator causes everyone to be worse-off. Braess' paradox is due to selfish users collectively optimising a different objective from the social journey time, so the addition of a link can possibly lead to worse social journey time. Our paradox, however, is due to the nontransivity of random variable comparisons where a better link in the global sense may not be the better one in the local sense.

In the full paper, we present other possible consequences of stochastic metrics such as sub-optimality in route selection and minimum spanning tree construction, divergence in routing policy, and oscillations in resource reservation, all of which are (disguised) forms of the cyclic preferences arising in the nontransitive dice game.

*A full paper is available at http://www.cl.cam.ac.uk/∼ckc25/dice